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FLOW OF A NEWTONIAN FLUID IN THE GAP BETWEEN CONICAL PERMEABLE SURFACES

V. M. Chervyakov and A. A. Koptev

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The equations of motion of a viscous Newtonian fluid between rotating and stationary conical coaxial permeable surfaces have been obtained from the Navier–Stokes differential equations and have been solved by the numerical method. The analytical dependences confirmed experimentally for determination of the energy consumption in rotary apparatuses have been obtained.

Investigation of the regularities of the motion of a Newtonian fluid between permeable surfaces is of scientific and applied importance. The efficiency of the processes in rotary atomizers [1], centrifugal granulators [2], and rotary apparatuses [3] is analyzed for flow bounded by porous walls. Furthermore, in some cases the main energy consumption of the apparatuses is dependent on the dissipative loss in the radial gaps between rotating permeable cylinders [3, 4]. We know of exact solutions for flow of a viscous incompressible fluid between rotating permeable cylinders [3, 5–7]. In some cases, the working surfaces of individual parts of chemical apparatuses represent conical coaxial surfaces. For example, in rotary apparatuses, they are made conical to control the size of the radial gap between the rotor and the stator [8]. There are a number of works in which flow of a liquid film in an impermeable rotating cone is considered [9, 10]. Therefore, it is important to theoretically investigate the regularities of the motion of the medium treated in the gap between permeable conical rotor and stator.

We consider a three-dimensional (axisymmetric about the z axis) problem of flow of a viscous fluid, i.e., $\partial/\partial\varphi = 0$. The differential Navier–Stokes equations of motion and continuity equations will be represented in a conical coordinate system ρ, φ, x (Fig. 1). The origin of the ρ axis is at the intersection of it and the z axis. This coordinate system is orthogonal. The Lamé coefficients for it are respectively equal to $H_\rho = 1$, $H_\varphi = \rho \sin\gamma + x\cos\gamma$, and $H_x = 1$ [11]. The equations of motion in the absolute coordinate system take the form

$$v_\rho \frac{\partial v_\rho}{\partial \rho} + v_x \frac{\partial v_\rho}{\partial x} - \frac{v_\varphi^2 \sin\gamma}{\rho\sin\gamma + x\cos\gamma} = -\frac{1}{\rho^*}\frac{\partial p}{\partial \rho} +$$

$$+ \nu\left[\frac{\partial^2 v_\rho}{\partial \rho^2} + \frac{\partial^2 v_\rho}{\partial x^2} - \frac{\dfrac{\partial v_\rho}{\partial \rho}\sin\gamma + \dfrac{\partial v_\rho}{\partial x}\cos\gamma}{\rho\sin\gamma + x\cos\gamma} - \frac{v_\rho \sin^2\gamma + v_x \sin\gamma\cos\gamma}{(\rho\sin\gamma + x\cos\gamma)^2}\right],$$

$$v_\rho \frac{\partial v_\varphi}{\partial \rho} + v_x \frac{\partial v_x}{\partial x} - \frac{v_\rho v_\varphi (\sin\gamma + \cos\gamma)}{\rho\sin\gamma + x\cos\gamma} = \tag{1}$$

$$= \nu\left[\frac{\partial^2 v_\varphi}{\partial \rho^2} + \frac{\partial^2 v_\varphi}{\partial x^2} - \frac{\dfrac{\partial v_\varphi}{\partial \rho}\sin\gamma + \dfrac{\partial v_\varphi}{\partial x}\cos\gamma}{\rho\sin\gamma + x\cos\gamma} - \frac{v_\varphi}{(\rho\sin\gamma + x\cos\gamma)^2}\right],$$

Tambov State Technical University, 106 Sovetskaya Str., Tambov, 392032, Russia; email: tmm-dm@mail.nnn.tstu.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 79, No. 2, pp. 92–98, March–April, 2006. Original article submitted November 9, 2004; revision submitted July 4, 2005.

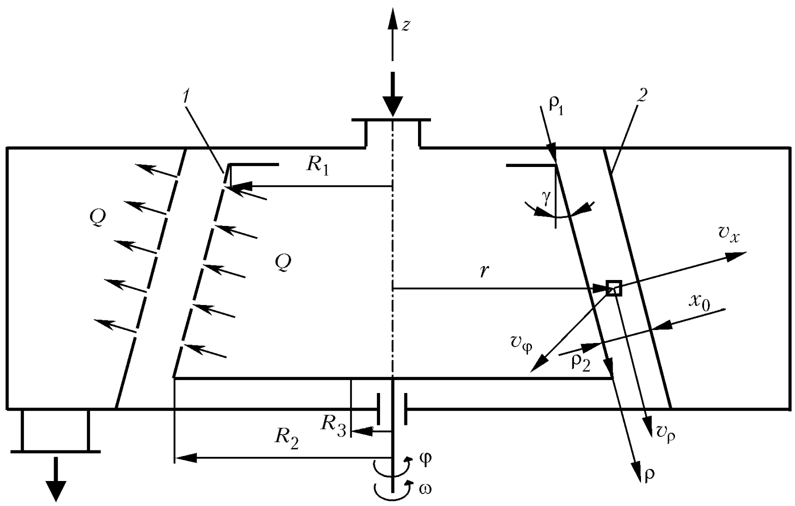

Fig. 1. Structural diagram of the rotor apparatus: 1) rotor; 2) stator.

$$v_\rho \frac{\partial v_x}{\partial \rho} + v_x \frac{\partial v_x}{\partial x} + \frac{v_\varphi^2 \cos\gamma}{\rho \sin\gamma + x\cos\gamma} = -\frac{1}{\rho^*}\frac{\partial p}{\partial x} +$$

$$+ \nu \left[\frac{\partial^2 v_x}{\partial \rho^2} + \frac{\partial^2 v_x}{\partial x^2} - \frac{\frac{\partial v_x}{\partial \rho}\sin\gamma + \frac{\partial v_x}{\partial x}\cos\gamma}{\rho\sin\gamma + x\cos\gamma} - \frac{v_\rho \sin\gamma\cos\gamma + v_x \cos^2\gamma}{(\rho\sin\gamma + x\cos\gamma)^2} \right].$$

The continuity equation for the incompressible fluid is

$$\frac{\partial v_\rho}{\partial \rho} + \frac{\partial v_x}{\partial x} + \frac{v_\rho \sin\gamma + v_x\cos\gamma}{\rho\sin\gamma + x\cos\gamma} = 0 . \quad (2)$$

In solving the problem posed, we make the following assumptions: the flow is steady-state laminar; the gravity forces are disregarded because of their smallness as compared to the centrifugal forces; the thickness of the gap x_0 is smaller than the change in the coordinate ρ. Therefore, in the Lamé coefficient for H_φ and in Eqs. (1) and (2), we take with a sufficient degree of accuracy that

$$\rho\sin\gamma + x\cos\gamma \approx \rho\sin\gamma . \quad (3)$$

To solve Eqs. (1) it is convenient to reduce them to dimensionless form using the following substitutions [12]:

$$v_\rho = \omega\rho\sin\gamma H'(\xi) , \quad v_\varphi = \omega\rho\sin\gamma G(\xi) , \quad v_x = -2\sqrt{\omega\rho\sin\gamma}H(\xi) .$$

$$p = -2\rho^*\omega\nu P(\xi) - C\omega^2\rho^2\sin^2\gamma/2 . \quad (4)$$

The relative coordinate ξ is determined as

$$\xi = x\sqrt{\frac{\omega\sin\gamma}{\nu}} . \quad (5)$$

The dimensionless velocity components H', G, and H are only functions of ξ; here, $H' = \partial H/\partial\xi$.

Substituting (4) and (5) into system (1) and taking account of (3), we obtain a system of the equations of motion of the fluid in the gap:

$$H''' = H'^2 - 2HH'' - G^2 + C\,,\quad G'' = 2H'G - 2HG'\,,\quad P' = H'' + 2HH'\,. \tag{6}$$

In deriving Eqs. (6), we have evaluated the significance of their terms with allowance for the numerical values of the parameters of actual rotor apparatuses ($x_0 \approx 10^{-4}$ m, $\gamma < \pi/18$, $\nu = 10^{-6}$ m^2/sec, and $\omega = 150–300$ sec^{-1}). Checking shows that the continuity condition (2) is fulfilled with a high accuracy.

Boundary conditions take the form: (a) on the exterior rotor surface for $x = 0$ and $\xi = 0$

$$H' = 0\,,\quad G = 1\,,\quad H = W\,, \tag{7}$$

b) on the interior stator surface for $x = x_0$ and $\xi = \xi_0$

$$H' = 0\,,\quad G = 0\,,\quad H = W\,. \tag{8}$$

In boundary conditions (7) and (8), we disregard the flow rate through the gap because of its smallness, i.e., we consider that the entire flow of the medium passes through the channels of the rotor and the stator. Therefore, we take

$$W = -\frac{v\big|_{x=0}}{2\sqrt{\omega\nu}\sin\gamma} = -\frac{v\big|_{x=x_0}}{2\sqrt{\omega\nu}\sin\gamma}\,; \tag{9}$$

here, $v\big|_{x=0} = v\big|_{x=x_0} = Q/S$. The lateral surfaces of the conical rotor and stator are determined, in view of the smallness of the radial gap, from the expression

$$S = \pi\,(\rho_2^2 - \rho_1^2)\sin\gamma\,.$$

The system of Eqs. (6) has no solution in closed form. For solution we represent the functions H and G as being expanded in power series in the vicinity of $\xi = 0$, i.e., in Taylor series

$$H = H + \frac{H'}{1!}\xi + \frac{H''}{2!}\xi^2 + \frac{H'''}{3!}\xi^3 + \ldots \qquad G = G + \frac{G'}{1!}\xi + \frac{G''}{2!}\xi^2 + \frac{G'''}{3!}\xi^3 + \ldots\,. \tag{10}$$

For determination of the values of all the derivatives and the functions themselves for $\xi = 0$, introducing the notation $H''(0) = A$ and $G'(0) = B$, we use the differential equations (6). To find the quantities A, B, and C we use boundary conditions (8) and, after certain transformations, obtain the system of equations

$$\begin{gathered}
H(\xi_0) = \frac{A}{2!}\xi^2 + \frac{C - 1 - 2WA}{3!}\xi^3 + \frac{-2B - 2W(C - 1 - 2WA)}{4!}\xi^4 + \\
+ \frac{-2B^2 + 8WB + 4W^2(C - 1 - 2WA)}{5!}\xi^5 \ldots = 0\,, \\
H'(\xi_0) = A\xi + \frac{C - 1 - 2WA}{2!}\xi^2 + \frac{-2B - 2W(C - 1 - 2WA)}{3!}\xi^3 + \\
+ \frac{-2B^2 + 8WB + 4W^2(C - 1 - 2WA)}{4!}\xi^4 \ldots = 0\,, \\
G(\xi_0) = 1 + B\xi - \frac{2WB}{2!}\xi^2 + \frac{2A + 4W^2B}{3!}\xi^3 + \frac{2(C - 1 - 4WA) + 2AB - 8W^3B}{4!}\xi^4 + \\
+ \frac{-4B + 4B(C - 1 - 2WA) - 8W(C - 1 - 3WA) - 4WB(A - 4W^3)}{5!}\xi^5 + \ldots = 0\,.
\end{gathered} \tag{11}$$

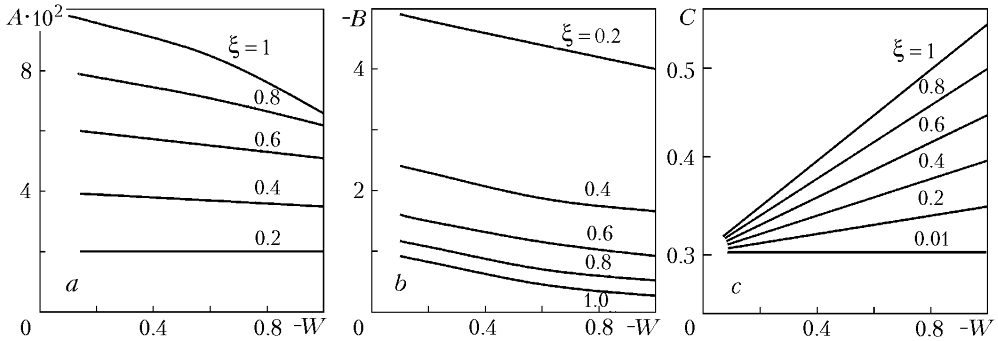

Fig. 2. Dimensionless second derivative of the velocity component normal to the cone surface ($A = G''$) (a), first derivative of the dimensionless circular velocity component ($B = G'$) (b), and constant C (c) vs. dimensionless coordinate ξ and dimensionless velocity component normal to the cone surface at entry into the gap and at exit from it W.

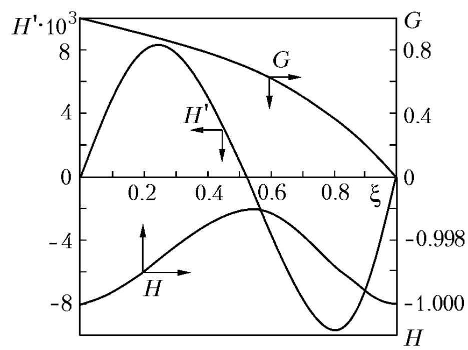

Fig. 3. Dimensionless velocity component normal to the cone surface H, meridian component H', and circular component G vs. dimensionless coordinate ξ ($\xi_0 = 1$ and $W = -1$).

We have used the numerical method of solution of Eqs. (11) to determine the dependences of A, B, and C on ξ_0 and W. It is noteworthy that the numerical values of the range of variation in ξ_0 and W have been selected based on the real values of the geometric and operating parameters of the rotor apparatus and the physicochemical properties of the medium treated. Figure 2 gives certain results of the numerical calculation. From an analysis of Fig. 2a and c, it can be inferred that the influence of the dimensionless component W normal to the cone surface at entry into the gap and consequently of the rate of flow of the treated medium through the apparatus on A and C increases with increase in the relative gap ξ_0. The virtually equidistant curves in Fig. 2b suggest that the influence of W on the derivative of the circular velocity $G'(\xi)$ is the same for all values of the dimensionless radial gap. It is noteworthy that, for $\xi_0 = 0.01$, we have $A \approx 10^{-3}$ in Fig. 2a and $B \approx -10$ in Fig. 2b. These values are not shown in the figures.

Figure 3 gives results of determination of the dependence of the meridian H', circular G, and normal-to-the cone surface H velocity components on the dimensionless gap ξ_0 and the velocity component normal to the cone surface at entry into the gap and at exit from it W. As follows from the physical ideas of the hydromechanical regularities of the process of flow in the gap, the character of variation (behavior) of the dependence $H(\xi)$ has an extremum approximately at the center of the gap between the rotor and the stator. This is logically explained by the influence of the meridian velocity component, but the variation in the numerical values of H is small for small radial gaps. It is noteworthy that this variation substantially increases with the gap H. The behavior of the dependence $H'(\xi)$ suggests that the directions of flow of the meridian component near the stator and the rotor are opposite. Near the rotating rotor, we have the outflow of the fluid from the gap along the generatrix of the rotor, approximately to the center of the gap, whereas near the stator we have the suction of the medium into the gap. The numerical values of H' substantially increase with gap. The behavior of the circular velocity G is the same for all ξ_0 values but, as the gap increases, the $G(\xi)$ curve has a character more convex relative to the ξ axis, which is quite consistent with the physical ideas of fluid flow in the gap between stationary and rotating surfaces.

Determining the velocity field by simultaneous solution of the first two equations of (6), we find the derivative of pressure $P'(\xi)$ and, after its integration, obtain the expression

$$P - P(0) = H' + H^2 ,$$

which enables us to determine a variation in the pressure in the gap between the rotor and the stator.

The theoretical investigations carried out enable us to pass to a more substantiated establishment of the power loss in the gap between the conical rotor and stator. The topicality of this problem stems from the fact that the main energy loss in apparatuses containing a rotating rotor and a stationary stator occurs in the gap between them [3, 13].

We determine the unit torque of frictional forces in the gap in the circular direction:

$$dM = \rho \sin \gamma dF . \tag{12}$$

According to the Newton law, the unit frictional force will be

$$dF = \mu \left| \frac{\partial v_\varphi}{\partial x} \right| dS , \tag{13}$$

and the area of action of the unit frictional force will be

$$dS = 2\pi\rho \sin \gamma d\rho . \tag{14}$$

To find the derivative of the circular velocity component we use the expressions for the circular velocity (second formula of (4) and expression (5) solved for x). Differentiating the expressions obtained, we have

$$\frac{\partial v_\varphi}{\partial x} = \omega\rho \sin \gamma^{1/2} \omega^{1/2} \nu^{-1/2} G'(\xi_0) . \tag{15}$$

Expressing the dynamic viscosity by the kinematic viscosity $\mu = \nu\rho^*$ and substituting (13) and (14) into (12), we obtain

$$dM = 2\pi\rho^* \omega^{3/2} \sin \gamma^{7/2} \nu^{1/2} \rho^3 G'(\xi_0)\ d\rho . \tag{16}$$

Integrating (16) in the limits from ρ_1 to ρ_2 with allowance for the fact that $r = \rho \sin \gamma$, we obtain the expression for the frictional torque:

$$M_{\text{fr}} = \frac{\pi}{2} (R_2^4 - R_1^4)\ \omega^{3/2} \nu^{1/2} \rho^* \sin \gamma^{-1/2}\ G'(\xi_0) . \tag{17}$$

The power dissipated in the radial gap between the conical rotor and stator is determined from the expression

$$N_{\text{rad.g}} = M\omega . \tag{18}$$

Substituting (17) into (18), finally we find

$$N_{\text{rad.g}} = \frac{\pi}{2} (R_2^4 - R_1^4)\ \omega^{5/2} \nu^{1/2} \rho^* \sin \gamma^{-1/2}\ G'(\xi_0) . \tag{19}$$

According to the notation adopted earlier, the value of the derivative of velocity $G'(\xi_0)$ is equal to B and is determined from the plots analogous to those in Fig. 2b or is calculated for each specific case.

Among the advantages of the flow model proposed is the fact that expression (6) can be used for finding the regularities of flow of a viscous fluid in the gap between stationary and rotating disks. For this purpose, we must take

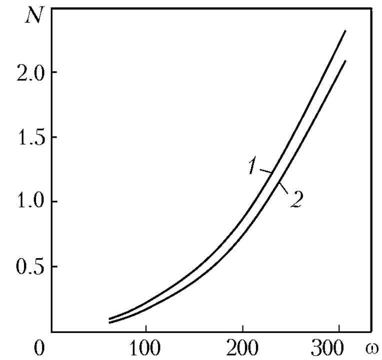

Fig. 4. Power consumption of the rotor apparatus (1) (experimental data of [14]) and consumption determined from dependence (27) (2) vs. angular rotational velocity. N, kW; ω, $\sec^{-1}$.

the angle $\gamma = \pi/2$ in the substitutions (4) and (5). This will be used for determination of the energy dissipation in the axial gap between the bottom of the rotor and the chamber (Fig. 1). Expression (19) will be written in the form

$$N_{\text{ax.g}} = \frac{\pi}{2} (R_2^4 - R_3^4)\, \omega^{5/2} \nu^{1/2} \rho^* G'(\xi_0) . \tag{20}$$

The value of $G'(\xi_0)$ in formula (20) is taken according to [12], where it has been determined on the surface of an impermeable rotating disk

$$G'(\xi_0) = 0.616 . \tag{21}$$

Thus, expression (20), with account for (21), has the form

$$N_{\text{ax.g}} = 0.308\, \pi\omega^{5/2} \nu^{1/2} \rho^* \,(R_2^4 - R_3^4) . \tag{22}$$

In the rotor apparatus, the power also goes into transferring kinetic energy to the fluid in the rotating rotor:

$$N_{\text{k}} = 0.5\rho^* Q\, (\omega^2 R_{\text{mean}}^2 + v\big|_{x=x_0}^2) , \tag{23}$$

the mean radius of the conical rotor is determined as

$$R_{\text{mean}} = (R_1 + R_2)/2 . \tag{24}$$

In the existing structures of rotor apparatuses, we usually have $\omega R_{\text{m}} >> v\Big|_{x=x_0}$ [8, 13, 14]; therefore, (23) is transformed, with a sufficient degree of accuracy, into the expression

$$N_{\text{k}} = 0.5\rho^* Q\omega^2 R_{\text{mean}}^2 . \tag{25}$$

In computing the power consumed by the rotor apparatus, we allow for the loss by friction in rotating parts of the structure; in fact, it is determined by the mechanical efficiency

$$N_{\text{mech}} = (0.05 - 0.07)\, (N_{\text{k}} + N_{\text{rad.g}} + N_{\text{ax.g}}) . \tag{26}$$

Finally, the energy consumption of the rotor apparatus will be represented as

$$N = N_{\text{k}} + N_{\text{ax.g}} + N_{\text{rad.g}} + N_{\text{mech}} . \tag{27}$$

To check the adequacy of the proposed model of flow of a medium in the gap between the conical rotor and stator, i.e., expression (27), in fact, we have used experimental data on finding the power consumed by a rotor apparatus [14]. The relative coordinate ξ_0 was determined from expression (5) for each angular rotational velocity of the rotor (0–300 sec^{-1}); water with $\rho^* = 10^3$ kg/m^3 and $\nu = 10^{-6}$ m^2/sec was used as the model fluid; the gap between the rotor and the stator was $x_0 = 10^{-4}$ m; the angle γ was equal to 6^o; the flow rate was $Q = 0.0014$ m^3/sec. Next we found the value of $G'(\xi_0)$ and computed the power dissipated in the gap from expression (19). Thereafter, we determined the value of the terms of Eq. (27) from dependences (22), (25), and (26). The results obtained are presented in Fig. 4. In it, it is seen that the experimental curve 1 lies above the theoretical curve 2 calculated from expression (27). The reason is that the loss by overcoming frictional forces on the interior rotor surface is disregarded in calculating the power. Nonetheless, taking into account the small disagreement between the calculated and experimental data (no more than 10%), we can assume that the flow model proposed and the procedure of determination of the power consumption are satisfactorily confirmed by experimental data.

We note that expressions (19), (22), and (25) do not contain experimental coefficients and exponents, as, for example, in [13, 14], and the quantities used in these expressions usually correspond to the requests for proposal of technological equipment. Thus, the procedure of determination of the power proposed can be used for tentative evaluation of the energy capacity (specific energy consumption) of the equipment in use having conical and plane surfaces.

The proposed three-dimensional model of flow of a Newtonian fluid in the gap between conical permeable surfaces makes it possible to improve the accuracy of calculations and to allow for the most important features of actual apparatuses: the influence of the size of the gap and the boundary flows. The model can be generalized to the case of flow of a non-Newtonian fluid.

NOTATION

$A = H''(0)$, second derivative of the dimensionless velocity component normal to the cone surface for the dimensionless coordinate equal to zero; $B = G'(0)$, first derivative of the dimensionless circular velocity for the dimensionless coordinate equal to zero; C, constant determined by integration of Eqs. (6) with specific boundary conditions; F, frictional force on the surface of a rotating cone, N; G, dimensionless circular velocity component; H, dimensionless velocity component normal to the cone surface; H_ρ, H_φ, and H_x, Lamé coefficients for the selected coordinate system; H', dimensionless meridian velocity component; M, frictional torque on the surface of a rotating cone, N·m; N, dissipated power, W; p, pressure, Pa; P and $P(0)$, dimensionless pressure and pressure at the origin of coordinates; Q, rate of flow of the fluid through the apparatus, m^3/sec; r, radial coordinate, cylindrical system, m; R, radius of the conical rotor, m; S, area of the lateral cone surface, m^2; v, velocity components on the coordinate axes, m/sec; W, dimensionless velocity component normal to the cone surface at exit from the gap and at entry into it; x, radial coordinate, conical system, m; z, vertical coordinate, cylindrical system, m; γ, slope of the lateral cone surface, rad; μ, dynamic coefficient of viscosity of the fluid, Pa·sec; ν, kinematic coefficient of viscosity of the fluid, m^2/sec; ξ, dimensionless radial coordinate, conical system; ρ, meridian coordinate, conical system, m; ρ^*, volume density of the fluid, kg/m^3; φ, circular coordinate, rad; ω, angular rotational velocity of the rotor, sec^{-1}. Subscripts: ax.g, axial gap; r.g, radial gap; k, kinetic energy; mech, mechanical; mean, mean; fr, friction; x, component on the x axis; φ, component on the φ axis; ρ, component on the ρ axis; 0, exterior surface of the conical rotor; 1, 2, and 3, small, large, and structural radii of the conical rotor respectively; $'$, $''$, and $'''$, derivatives with respect to ξ.

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